Limits and Continuity

Let $y = f(x)$ be a function. Suppose that a and **L** are numbers such that:

- whenever x is close to a but not equal to a, $f(x)$ is close to **L**;
- as x gets closer and closer to a but not equal to a, $f(x)$ gets closer and closer to **L**; and
- suppose that $f(x)$ can be made as close as we want to **L** by making x close to a but not equal to a.

Then we say that **the limit of $f(x)$ as x approaches a is L** and we write $\lim_{x \to a} f(x) = L$

As an example, consider the function $g(x) = 3x-2$. We see that $g(7) = 19$ and we ask: if x is close to 7, is $g(x)$ close to 19?
So, we can see that if x is close to 7, then $g(x)$ approaches 19. In this case, a is 7 and L is 19.

Consider the following limit: $\lim_{x \to 2} \frac{x^2 - 3x}{4x - 3}$

By substituting $x = 2$, we get $\frac{2^2 - 6}{8 - 3} = \frac{-2}{11}$

Q Is that all there is to evaluating limits algebraically: just substitute the number x is approaching in the given expression?

A Not always, but this often does happen, and when it does, we say that the function is **continuous** at the value of x in question.

Look at another example: $\lim_{x \to 5} \frac{\sqrt{5x - 3}}{x - 3}$

Substituting $x = 5$ gives $\frac{\sqrt{25} - 3}{5 - 3} = 1$

Evaluating a Limit Using Simplification

Let us evaluate $\lim_{x \to -2} \frac{3x^2 + x - 10}{x + 2}$

Ask yourself the following questions:

1. Is the function $f(x)$ a polynomial function?

Answer: No, but the numerator and denominator separately are polynomials. They are combined into a single mathematical formula and so are now a closed-form function.

2. Is the value $x = a$ in the domain of $f(x)$?

Answer: No, since $(3(-2)^2+(-2)-10)/((-2)+2)$ is not defined.

Therefore, we consult the above Question/Answer discussion, and simplify the function, if we can.

$$\frac{3x^2 + x - 10}{x + 2} = \frac{(x + 2)\,(3x - 5)}{(x + 2)} = 3x - 5$$

Since we are now left with a polynomial function that is defined when $x = -2$, we can now evaluate the limit by substitution:

$$\text{Lim}_{x \to -2} \frac{3x^2 + x - 10}{x + 2} = \lim_{x \to -2} 3x - 5 = 3(-2) - 5 = -11$$

In addition to these, there are some limit laws that should be understood. These are given below:

LIMIT LAWS Suppose c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then:

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$
4. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

Q How do you tell if a function is continuous?

A Sometimes, it is easy: A polynomial function is always continuous. Rational functions with non-zero denominators as well as the sine and cosine functions are also continuous. Other continuous functions include root functions, exponential functions, and logarithmic functions.

Continuous Functions

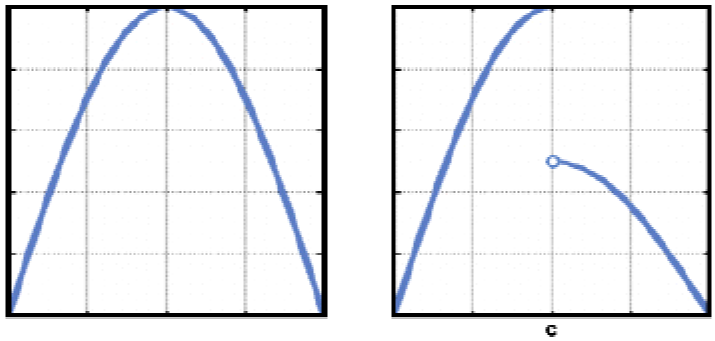
The function $f(x)$ is **continuous** at $x = a$ if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) \text{ i.e. } \lim_{x\to a} f(x) \text{ exists and equals } f(a)$$

The function f is said to be **continuous on its domain** if it is continuous at each point in its domain. If f is not continuous at a particular value 'a,' we say that f is **discontinuous** at a or that f has a **discontinuity** at a.

Limits and Continuity

Intuitively, a function is continuous if you can draw it without lifting your pen from your paper. In the diagram below, the function the function on the left is continuous throughout, but the function on the right is not. It is "discontinuous" at $x = c$.



Q Is $f(x) = x^3 + 2x + 1$ continuous at $x = 2$?

A Yes. All polynomial functions are continuous. To check, $\lim_{x \to 2} f(x) = 13$ and $f(2) = 13$.

Q Is $f(x) = \frac{x^2 - 16}{x - 4}$ continuous at $x = 4$?

A No. $f(4)$ is undefined.

Q Is the following function continuous? $f(x) = \begin{cases} x + 3 \; for \; x < 2 \\ 5 \; for \; x = 2 \\ x^4 - 11 \; for \; x > 2 \end{cases}$

A Since this function is a junction of two continuous functions, we only have to worry about discontinuity at the point where the functions meet, i.e. at $x = 2$.

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x + 3 = 2 + 3 = 5$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^4 - 11) = 2^4 - 11 = 5$

Thus, $\lim_{x \to 2} f(x) = 5$, and $f(2) = 5$, so $\lim_{x \to 2} f(x) = f(2)$. It follows that f is continuous.

Q Is there a way to define $f(c)$ for the for the following function so that $f(x)$ is continuous at $x = c$?

$f(x) = \frac{x^2 - 16}{x - 4}$ at $c = 4$

A

$\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{x^2 - 16}{x - 4}$

$= \lim_{x \to 4} \left(\frac{(x+4)(x-4)}{x-4}\right)$

$= \lim_{x \to 4} (x + 4)$

$= 8$

Because $f(x) = 8$, we should define $\lim_{x \to 4} f(4) = 8$ to make this function continuous.