Optimization- What is the Minimum or Maximum?

Here is an application of calculus (finally…) that is utilized by many in their daily lives. Say, I have some amount of fencing and I want to find out the dimensions that would give me the largest area? Its Christmas time and I have to make a gift box. I have some paper that I could fold to make the box. Again, what dimensions do I use to maximize volume? As a business manager one often asks questions about how one can minimize costs?

The answers to all these questions lie in *Optimization*.

Here are a few steps to solve optimization problems:

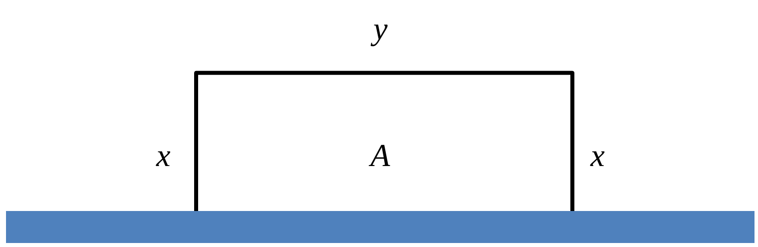
1. Read the problem- write the knowns, unknowns, and draw a diagram if applicable.
2. Write down an equation for what needs to be maximized/minimized (such as A=b*h or Cost= (price)*(number of units) etc.)
3. Write the function in step 2 in terms of one variable by using a giving relationship from step 1 i.e. write A= in terms of base only or cost = f(price) only (this step become clear in the following example)
4. Now find the derivative of the function found in step 3
5. Find the critical values (i.e. set the derivative = 0 and solve for the variable)
6. The critical value obtained is either the maximum or minimum (check for yourself on the number line). Find the other variable.

In order to understand this further, let’s solve a sample problem.

Q. A farmer has 2400 ft. of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

1. *Read the problem- write the knowns, unknowns, and draw a diagram if applicable*

Perimeter of fencing = 2400
Area = ?



2. *Write down an equation for what needs to be maximized/minimized*

A= base*height = xy

3. *Write the function in step 2 terms of one variable by using a giving relationship from step___*

We know that the perimeter of fence = 2400. In our case that means 2x + y = 2400. This tells us

$$y = 2400-2x$$

Therefore area can be written as $A= x(2400-2x) = 2400 x -2x^2$

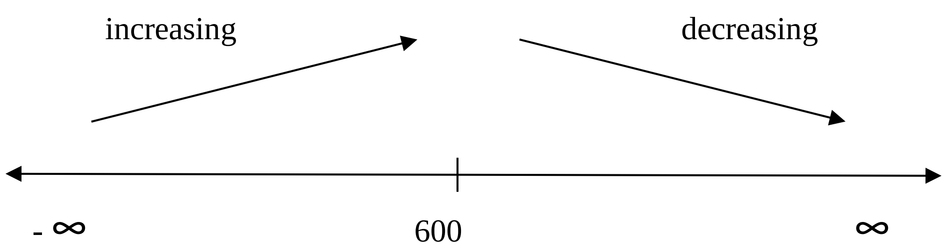
4. *Now find the derivative of the function found in step 3*

$$A' = 2400-4x$$

5. *Find the derivate of the function found in step 3*

$$2400 - 4x = 0$$
$$x= 600$$

6. *The critical value obtained is either the maximum or minimum. Find the other variable.*



$A'(x) = 2400-4x$
$A' (500) = 2400-4(500) = 400$
$A' (700) = 2400- 4(700) = -400$

If width (x) = 600 feet,

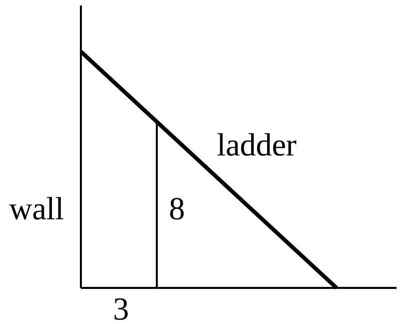
then length (y) = 2400- 2x = 2400 – 1200 = 1200 feet

*Thus the rectangular field should be **600 feet** wide and **1200 feet** long.*

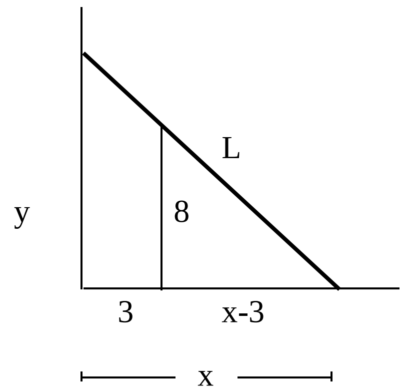
Optimization- What is the Minimum or Maximum?

Let's try another problem

Q. Find the length of the shortest ladder that will reach over an 8-ft. high fence to a large wall which is 3 ft. behind the fence. (See diagram.)



1. *Read the problem- write the knowns, unknowns and draw a diagram if applicable*



2. *Write down an equation for what needs to be maximized/minimized*

I need to minimize length of the ladder 'L'

Using Pythagorean Theorem, I can say that L = $\sqrt{x^2 + y^2}$

3. *Write the function in step 2 in terms of one variable by using a giving relationship from step ____*

I can use similar triangles to derive a relationship between x and y

$$\frac{y}{x} = \frac{8}{x-3}$$

So,

$$\frac{y}{} = \frac{8}{x-3}$$

Optimization- What is the Minimum or Maximum?

$$\mathbf{L} = \sqrt{x^2 + \left(\frac{8x}{x-3}\right)^2}$$

$$= \sqrt{x^2 + \frac{64x^2}{(x-3)^2}}$$

4. *Now find the derivative of the function found in step 3*

$$L' = (1/2)\left(x^2 + \frac{64x^2}{(x-3)^2}\right)^{-1/2} \left\{2x + \frac{64x(x-3)^2(128x)-64x^2)2(x-3)}{(x-3)^4}\right\}$$

5. *Find the critical values*

(Factor out 64x and (x-3) from the numerator of the fraction inside the brackets)

$$=(1/2)\left(x^2 + \frac{64x^2}{(x-3)^2}\right)^{-1/2}\left\{2x + \frac{64x(x-3)[2(x-3)2x]}{(x-3)^4}\right\}$$

(Divide out a factor of (x-3) and simplify the entire expression)

$$\frac{2x + \frac{64x(-6)}{(x-3)^3}}{\sqrt[2]{x^2 + \frac{64x^2}{(x-3)^2}}}$$

(Factor out 2x from the numerator)

$$= \frac{2x\{1 + \frac{-192}{(x-3)^3}}{\sqrt[2]{x^2 + \frac{64x^2}{(x-3)^2}}}$$

$$\frac{x\{1 + \frac{-192}{(x-3)^3}}{\sqrt[2]{x^2 + \frac{64x^2}{(x-3)^2}}}$$

$$= 0,$$

$$\text{i.e. } x\left\{1 + \frac{-192}{(x-3)^3}\right\} = 0.$$

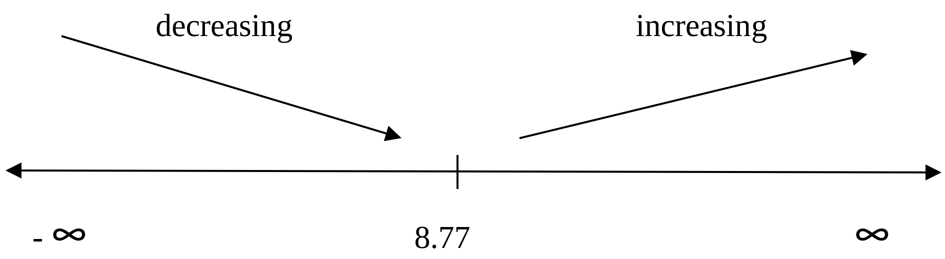
$$x = 0$$

or

$$1-\frac{192}{(x-3)^3}=0,$$

x= 8.77

6. *The critical value obtained is either the maximum or minimum. Find the other variable.*



If x = 8.77 feet,
then y= 8(8.77)/(8-3) = 16.67 feet

Thus the minimum length of the ladder is $\sqrt{8.77^2 + 16.67^2}$= ***17.64 feet***

Optimization- What is the Minimum or Maximum?

Try some sample problems

1) What is the largest rectangular area that 80 feet of fencing can enclose?

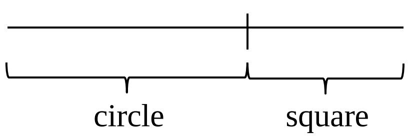
2) A rectangle has one side on the x-axis and two vertices on the curve $y = \sqrt{1 - x^2}$. What is the maximum area such a rectangle can have? The minimum area?

3) A landscape architect plans to enclose a 5000 square foot rectangular region in a botanical garden. She will use shrubs costing \$25 per foot along one side and fencing costing \$10 per foot along the remaining four sides. Find the dimensions of the enclosure which minimizes the total cost of the construction (round your answer to the nearest foot).

4) The speed of traffic through the Lincoln Tunnel depends on the density of the traffic. Let S be the speed in miles per hour and D be the density in vehicles per mile. The relationship between S and D is approximately $S = 42 - \frac{D}{3}$ for $D \leq 100$. Find the density that will maximize the hourly flow.

5) The Can-O-Rad Company manufactures cylindrical barrels to store nuclear waste. The top and bottom of the barrels are to be made with material that costs \$10 per square foot and the rest is made with material that costs \$8 per square foot. If each barrel is to hold 5 cubic feet, find the dimensions of the barrel that will minimize the total cost.

6) A wire of length 12 inches can be bent into a circle, a square, or cut to make both a circle and a square. How much wire should be used for the circle if the total area enclosed by the figure(s) is to be a minimum? A maximum?



7) A window consisting of a rectangular topped by a semicircle is to have a perimeter P. find the radius of the semicircle if the area of the window is to be a maximum.

8) A furniture business rents chairs for conferences. A contract is drawn to rent and deliver up to 400 chairs for a particular meeting. The exact number would be determined by the customer later. The price will be \$90 per chair up to 300 chairs. If the order goes above 300 chairs, the price would be reduced by \$0.25 per chair for every additional chair ordered above 300. This reduced price would be applied to the entire order. Determine the largest and smallest revenues this business can make under this contract.

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